

On the solvability of forward-backward stochastic differential equations driven by Teugels Martingales

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Abstract

We deal with a class of fully coupled forward-backward stochastic differential equations (FBSDE for short), driven by Teugels martingales associated with some Lévy process. Under some assumptions on the derivatives of the coefficients, we prove the existence and uniqueness of a global solution on an arbitrarily large time interval. Moreover, we establish stability and comparison theorems for the solutions of such equations. Note that the present work extends known results by Jianfeng Zhang (Discrete Contin. Dyn. Syst. Ser. B 6 (2006), no. 4, 927–940), proved for FBSDEs driven by a Brownian motion, to FBSDEs driven by general Lévy processes.

Keywords: Forward-backward stochastic differential equations; Teugels Martingale; Lévy process.

1 Introduction

Let $(L_t)_{0 \leq t \leq T}$ be a \mathbb{R} -valued Lévy process defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions. Assume that the Lévy measure $\nu(dz)$ corresponding to the Lévy process L_t satisfies:

- (i) $\int_{\mathbb{R}} (1 \wedge z^2) \nu(dz) < \infty$,
- (ii) there exist $\alpha > 0$ such that for every $\varepsilon > 0$,

$$\int_{]-\varepsilon, \varepsilon[^c} e^{\alpha|z|} \nu(dz) < \infty.$$

Assumptions (i) and (ii) imply in particular that the random variable $L(t)$ has moments of all orders. We also assume that $\mathcal{F}_t = \mathcal{F}_0 \vee \sigma(L_s, s \leq t) \vee \mathcal{N}$, where $\mathcal{G}_1 \vee \mathcal{G}_2$ denotes the σ -field generated by $\mathcal{G}_1 \cup \mathcal{G}_2$ and \mathcal{N} is the totality of the P -negligible sets.

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The aim of this work is to prove existence and uniqueness of solutions of the following coupled forward-backward stochastic differential equation (FBSDE for short)

$$\begin{cases} X_t = X_0 + \int_0^t f(s, w, X_s, Y_s, Z_s) ds + \sum_{i=1}^{\infty} \int_0^t \sigma^i(s, w, X_{s-}, Y_{s-}) dH_s^i, \\ Y_t = \varphi(X_T) + \int_t^T g(s, w, X_s, Y_s, Z_s) ds - \sum_{i=1}^{\infty} \int_t^T Z_s^i dH_s^i, \end{cases} \quad (1.1)$$

where $t \in [0, T]$, $H_t = (H_t^i)_{i=1}^{\infty}$ are pairwise strongly orthonormal Teugels martingales associated with the Lévy process L_t . For any \mathbb{R} -valued and \mathcal{F}_0 -measurable random vector X_0 , satisfying $\mathbb{E}|X_0|^2 < \infty$, we are looking for an $\mathbb{R} \times \mathbb{R} \times l(\mathbb{R})$ -valued solution (X_t, Y_t, Z_t) on an arbitrarily fixed large time duration, which is square-integrable and adapted with respect to the filtration \mathcal{F}_t generated by L_t and \mathcal{F}_0 satisfying

$$\mathbb{E} \int_0^t (|X_t|^2 + |Y_t|^2 + |Z_t|^2) dt < \infty.$$

The existence and uniqueness of solutions of FBSDEs without the Teugels part have been widely studied by many authors (see, e.g. [1], [4], [6], [7], [10], [11], and [15]). The first study of FBSDEs has been performed by Antonelli [1] in the early 1990s. The author has used the contraction mapping technique to obtain a local existence and uniqueness result in a small time interval. Hu and Peng [6] have used a probabilistic method to establish an existence and uniqueness result, under certain monotonicity conditions, in the case where the forward and backward components have the same dimension. Then Hamadène [5] improved their result by proving it under weaker monotonicity assumptions. Peng and Wu provided in [11] more general results by extending the two above results, without the restriction on the dimensions of the forward and backward parts.

In spite of the large literature devoted to the Brownian case as we have mentioned above, there are relatively a few results on FBSDEs driven by Teugels Martingales. To the best of our knowledge, the first paper dealing with this kind of equations driven by Lévy processes is [12], where the authors have proved the existence and uniqueness via the solution of its associated partial integro-differential equation (PIDE for short). Then Bagheri et al. [2] proved under some monotonicity assumptions, the existence and uniqueness of solutions on an arbitrarily fixed large time duration.

Motivated by the above results and by imposing an assumption on the derivatives of the coefficients, introduced by Zhang [16], we establish two main results. We shall first prove the existence and uniqueness of the solution of the FBSDE 1.1, without any restriction on the time duration. The main idea of the proof is to construct the solution on small intervals, and then extend it piece by piece to the whole interval. In a second step, we prove stability and comparison theorems for the solutions. Let us point out that our work extends the results of Jianfeng Zhang (Discrete Contin. Dyn. Syst. Ser. B 6 (2006), no. 4, 927–940), to FBSDEs driven by general Lévy processes. We note that much of the technical difficulties coming from the Teugels martingales are due to the fact that the quadratic variation $[H^i, H^j]$ is not absolutely continuous, with respect to the Lebesgue measure. To overcome these difficulties, we use the fact that the predictable quadratic variation process $\langle H^i, H^j \rangle_t$ is equal to $\delta_{ij}t$ and that $[H^i, H^j]_t - \langle H^i, H^j \rangle_t$ is a martingale.

This paper is organized as follows. In Section 2, we give some preliminaries and notations about Teugels martingales. In Section 3, we give some assumptions and provide our main results. The proofs are provided in the last section.

2 Notations and assumptions

Let us recall briefly the L^2 theory of Lévy processes as it is investigated in Nualart-Schoutens [8]. A convenient basis for martingale representation is provided by the so-called Teugels martingales. This means that this family has the predictable representation property.

Denote by $\Delta L_t = L_t - L_{t-}$ where

$$L_{t-} = \lim_{s \rightarrow t, s < t} L_s, \quad t > 0,$$

and define the power jump processes by

$$L_t^{(i)} = \begin{cases} L_t & \text{if } i = 1; \\ \sum_{0 < s \leq t} (\Delta L_s)^i & \text{if } i \geq 2. \end{cases}$$

If we denote

$$Y_t^{(i)} = L_t^{(i)} - \mathbb{E} [L_t^{(i)}], \quad i \geq 1,$$

with

$$\mathbb{E} [L_t^{(1)}] = \mathbb{E} [L_t] = t\mathbb{E} [L_1] = tm_1,$$

and, for $i \geq 2$

$$\mathbb{E} [L_t^{(i)}] = \mathbb{E} \left[\sum_{0 < s \leq t} (\Delta L(s))^i \right] = t \int_{-\infty}^{\infty} z^i \nu(dz) = tm_i.$$

Then the family of Teugels martingales $(H_t^i)_{i=1}^{\infty}$, is defined by

$$H_t^i = \sum_{j=1}^{j=i} a_{ij} Y_t^{(j)}.$$

The coefficients a_{ij} correspond to the orthonormalization of the polynomials $1, x, x^2, \dots$ with respect to the measure $\mu(dx) = x^2 \nu(dx) + \delta_0(dx)$. Then $(H_t^i)_{i=1}^{\infty}$ is a family of strongly orthogonal martingales such that $\langle H^i, H^j \rangle_t = \delta_{ij} \cdot t$ and $[H^i, H^j] - \langle H^i, H^j \rangle_t$ is a martingale, see [8, 13].

The following lemma which gives some useful properties of the Teugels martingale will be needed in the sequel.

Lemma 2.1. *i) The process H_t^i can be represented as follows:*

$$H_t^i = q_{i-1}(0) B_t + \int_{\mathbb{R}} p_i(x) \tilde{N}(t, dx)$$

where B_t be a Brownian motion, and $\tilde{N}(t, dx)$ is the compensated Poisson random measure that corresponds to the pure jump part of L_t and the polynomials $q_{i-1}(0)$ and $p_i(x)$ associated to L_t .

ii) The polynomials p_i and q_j are linked by the relation:

$$\int_{\mathbb{R}} p_i(x) p_j(x) v(dx) = \delta_{ij} - q_{i-1}(0) q_{j-1}(0).$$

Proof. See [12]. □

In the rest of this section, we list all the notations that will be frequently used throughout this work.

l^2 : the Hilbert space of real-valued sequences $x = (x_n)_{n \geq 0}$ with norm

$$\|x\| = \left(\sum_{i=1}^{\infty} x_i^2 \right)^{\frac{1}{2}} < \infty.$$

Let us define

$l^2(\mathbb{R})$: the space of \mathbb{R} -valued process $\{f^i\}_{i \geq 0}$ such that

$$\left(\sum_{i=1}^{\infty} \|f^i\|_{\mathbb{R}}^2 \right)^{\frac{1}{2}} < \infty.$$

$l^2_{\mathcal{F}}(0, T, \mathbb{R})$: the Banach space of $l^2(\mathbb{R})$ -valued \mathcal{F}_t -predictable processes such that

$$\left(\mathbb{E} \int_0^T \sum_{i=1}^{\infty} \|f^i(t)\|_{\mathbb{R}}^2 dt \right)^{\frac{1}{2}} < \infty.$$

$\mathcal{S}_{\mathcal{F}}^2(0, T, \mathbb{R})$: the Banach space of \mathbb{R} -valued \mathcal{F}_t -adapted and càdlàg processes such that

$$\left(\mathbb{E} \sup_{0 \leq t \leq T} |f(t)|^2 \right)^{\frac{1}{2}} < \infty.$$

$L^2(\Omega, \mathcal{F}, P, \mathbb{R})$: the Banach space of \mathbb{R} -valued, square integrable random variables on (Ω, \mathcal{F}, P) . Here and in what follows, for notational simplicity, we shall denote

$$\int_0^t \sigma(s, w, X_{s-}, Y_{s-}) dH_s \quad \text{and} \quad \int_t^T Z_s dH_s$$

instead of

$$\sum_{i=1}^{\infty} \int_0^t \sigma^i(s, w, X_{s-}, Y_{s-}) dH_s^i \quad \text{and} \quad \sum_{i=1}^{\infty} \int_t^T Z_s^i dH_s^i$$

respectively, where $Z_s = \{Z_s^i\}_{i=1}^{\infty}$, $\sigma_s = \{\sigma_s^i\}_{i=1}^{\infty}$, $\sigma^i : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow l^2(\mathbb{R})$. Further, for the notational simplicity, we have suppressed w and we will do so below. We also use the following notation

$$M^2(0, T) = \mathcal{S}_{\mathcal{F}}^2(0, T, \mathbb{R}) \times \mathcal{S}_{\mathcal{F}}^2(0, T, \mathbb{R}) \times l^2_{\mathcal{F}}(0, T, \mathbb{R}).$$

The following assumptions will be considered in this paper.

We suppose that the coefficients

$$\begin{aligned} f &: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times l^2(\mathbb{R}) \rightarrow \mathbb{R}, \\ \sigma &: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow l^2(\mathbb{R}), \\ g &: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times l^2(\mathbb{R}) \rightarrow \mathbb{R}, \\ \varphi &: \Omega \times \mathbb{R} \rightarrow \mathbb{R}, \end{aligned}$$

are progressively measurable, such that:

(**H₁**) There exist $\lambda, \lambda_0 > 0$, such that $\forall t \in [0, T], \forall (x, y, z)$ and (x', y', z') in $\mathbb{R} \times \mathbb{R} \times l(\mathbb{R})$

$$\begin{aligned} |f(t, x, y, z) - f(t, x', y', z')| &\leq \lambda \left(|x - x'| + |y - y'| + \|z - z'\|_{l^2(\mathbb{R})} \right), \\ |\sigma(t, x, y) - \sigma(t, x', y')|^2 &\leq \lambda^2 \left(|x - x'|^2 + |y - y'|^2 \right), \\ |g(t, x, y, z) - g(t, x', y', z')| &\leq \lambda \left(|x - x'| + |y - y'| + \|z - z'\|_{l^2(\mathbb{R})} \right), \\ |\varphi(x) - \varphi(x')| &\leq \lambda_0 (|x - x'|). \end{aligned}$$

(**H₂**) The functions f, g, σ, φ are differentiable with respect to x, y, z with uniformly bounded derivatives such that

$$\sigma_y f_z = 0 \text{ and } f_y + \sigma_x f_z + \sigma_y g_z = 0. \quad (2.1)$$

Let us mention that assumption (**H₂**) has been introduced bfor the first time by Zhang [16] in the case of FBSDEs without jumps.

3 The main results

3.1 Existence and uniqueness

The following theorem gives the existence of a solution in a small time duration.

Theorem 3.1. *Suppose that (**H₁**) is satisfied. Assume further that*

$$V_0^2 \triangleq \mathbb{E} \left\{ |X_0|^2 + |\varphi(0)|^2 + \int_0^T \left[|f(t, 0, 0, 0)|^2 + \|\sigma(t, 0, 0)\|_{l^2(\mathbb{R})}^2 + |g(t, 0, 0, 0)|^2 \right] dt \right\} < \infty.$$

Then, for every \mathcal{F}_0 -measurable random vector X_0 , there exists a constant δ depending only on λ and λ_0 , such that for $T \leq \delta$, equation (1.1) has a unique solution which belongs to $M^2(0, T)$.

The following proposition gives a priori estimates, which shows in particular the continuous dependence of the solution upon the data.

Proposition 3.1. *Under the same assumptions of the Theorem 3.1, there exist δ and C_0 depending on λ and λ_0 , such that for $T \leq \delta$, the following estimates hold true:*

i)

$$\|\Pi\| = \mathbb{E} \left(\sup_{0 \leq t \leq T} [|X_t|^2 + |Y_t|^2] + \int_0^T \|Z_t\|_{l^2(\mathbb{R})}^2 dt \right)^{\frac{1}{2}} \leq C_0 V_0.$$

ii)

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} [|X_t|^{2p} + |Y_t|^{2p}] + \left(\int_0^T \|Z_t\|_{l^2(\mathbb{R})}^2 dt \right)^p \right\} < \infty$$

The next Theorem extends the result in Theorem 3.1 to arbitrary large time duration.

Theorem 3.2. *Assume (\mathbf{H}_1) , (\mathbf{H}_2) and $V_0^2 < \infty$. Then:*

- i) *Equation (1.1) has a unique solution $\Pi \in M^2(0, T)$.*
- ii) *The following estimate holds*

$$\|\Pi\|^2 \leq CV_0^2.$$

3.2 Stability theorem

The following results state the stability of the solution of FBSDE (1.1) with respect to the initial condition and the data. This means that the solution of equation (1.1) does not change too much under small perturbations of the data. In other words, the trajectories which are close to each other at specific instant should therefore remain close to each other at all subsequent instants. To state the next theorem and its corollary, let us consider $\Pi^i, i = 0, 1$ the solutions of (1.1) corresponding to $(f^i, \sigma^i, g^i, \varphi^i)$. We shall consider the following notations, $\Delta\Pi \triangleq \Pi^1 - \Pi^0$ and for any function $h \triangleq f, \sigma, g, \varphi$, we set $\Delta h \triangleq h^1 - h^0$.

Theorem 3.3. *Assume that $(f^i, \sigma^i, g^i, \varphi^i, X_0^i), i = 0, 1$, satisfy the same conditions of Theorem 3.2. Then*

$$\|\Delta\Pi\|^2 \leq C\mathbb{E} \left\{ |\Delta X_0|^2 + |\Delta\varphi(X_T^1)|^2 + \int_0^T \left[|\Delta f|^2 + \|\Delta\sigma\|_{l^2(\mathbb{R})}^2 + |\Delta g|^2 \right] (t, \Pi_t^1) dt \right\}.$$

Corollary 3.1. *Suppose that $(f^n, \sigma^n, \varphi^n, g^n, X_0^n)$, for $n = 0, 1, \dots$ satisfy the same conditions of Theorem 3.2. Moreover assume that:*

- i) $X_0^n \rightarrow X_0^0$ in L^2 .
- ii) for $h \triangleq f, \sigma, \varphi, g$, $h^n(t, \Pi) \rightarrow h^0(t, \Pi)$ as $n \rightarrow \infty$.
- iii) $\mathbb{E} \left\{ |X_0^n - X_0^0|^2 + |\varphi^n - \varphi^0|^2(0) + \int_0^T \left[|f^n - f^0|^2 + \|\sigma^n - \sigma^0\|_{l^2(\mathbb{R})}^2 + |g^n - g^0|^2 \right] (t, 0, 0, 0) dt \right\} \rightarrow 0$

Then if Π^n (resp. Π) denotes the solution of (1.1) corresponding to $(f^n, \sigma^n, \varphi^n, g^n, X_0^n)$ (resp. $(f, \sigma, \varphi, g, X_0^0)$), we obtain

$$\|\Pi^n - \Pi^0\| \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

3.3 Comparison theorem

In what follows we provide, under the same assumptions as for the existence and uniqueness results, another important result, which is the comparison theorem. Let (X, Y, Z) be the solution to the following LFBSDE:

$$\begin{cases} X_t = \int_0^t (a_s^1 X_s + b_s^1 Y_s + c_s^1 Z_s) ds + \int_0^t (a_s^2 X_s + b_s^2 Y_s) dH_s, \\ Y_t = PX_T + \alpha + \int_t^T (a_s^3 X_s + b_s^3 Y_s + c_s^3 Z_s + \beta_s) ds - \int_t^T Z_s dH_s. \end{cases} \quad (3.1)$$

Then we have the following proposition, which is the linear version of the next theorem.

Proposition 3.2. Assume $|a_t^i|, |b_t^i|, |c_t^i| \leq \lambda$, $|P| \leq \lambda_0$ and (\mathbf{H}_2) holds true. Assume further that $\alpha \geq 0$ and $\beta_s \geq 0$. Then

$$Y_0 \geq 0.$$

Further we have the following general result. Let $\Pi^i, i = 0, 1$, be the solution of the following FBSDE:

$$\begin{cases} X_t^i = X_0 + \int_0^t f(s, \Pi_s^i) ds + \int_0^t \sigma(s, X_{s-}^i, Y_{s-}^i) dH_s, \\ Y_t^i = \varphi^i(X_T^i) + \int_t^T g^i(s, \Pi_s^i) ds - \int_t^T Z_s^i dH_s, i = 0, 1 \end{cases} \quad (3.2)$$

Theorem 3.4. Let $\Pi^i, i = 0, 1$, be the solutions of the FBSDEs (1.1). If

- i) $(f, \sigma, g^i, \varphi^i), i = 0, 1$ satisfy (\mathbf{H}_2) and $V_0^2 < \infty$.
- ii) For any $(t, \Pi), \varphi^0(X) \leq \varphi^1(X)$ and $g^0(t, \Pi) \leq g^1(t, \Pi)$. Then

$$Y_0^0 \leq Y_0^1.$$

We would like to mention that the above comparison theorem holds true only at time $t = 0$. We cannot get the result in the whole interval $[0, T]$, even in the Brownian case. See for instance, the counterexample which is given in [14].

Remark 3.1. We should point out that the following cases are in fact, involved in our present study.

1. **FBSDEs driven by Brownian motion:** If $\nu = 0$, then all non-zero degree polynomials $q_{i-1}(x)$ will vanish, $H_t^{(1)} = W_t$ is a standard Brownian motion and $H_t^{(i)} = 0$, for $i \geq 2$.
2. **FBSDEs driven by Poisson Process:** assume that μ only has mass at 1, then $H_t^{(i)} = N_t - \lambda t$ is the compensated Poisson process with intensity λ and also $H_t^{(i)} = 0$, for $i \geq 2$. For example, If we have $\nu(dx) = \sum_{j=1}^{\infty} \alpha_j \delta_{\beta_j}(dx)$, where $\delta_{\beta_j}(dx)$ denotes the positive mass measure at $\beta_j \in \mathbb{R}$ of size 1. Then, The process L . takes the form

$$L_t = at + \sum_{j=1}^{\infty} \left(N_t^{(j)} + \alpha_j t \right),$$

where $\{N_t^{(j)}\}_{j=1}^{+\infty}$ denote the sequence of independent Poisson process with parameters $\{\alpha_j\}_{j=1}^{+\infty}$. In this case

$$H_t^{(1)} = \sum_{j=1}^{\infty} \frac{\beta_1}{\sqrt{\alpha_j}} \left(N_t^{(j)} + \alpha_j t \right)$$

4 Proofs and technical results

4.1 Small time duration

In this subsection, we shall start by giving and proving the following technical Lemma, which will be used in the proof of Theorem 3.1. Let us introduce the following decoupled FBSDE:

$$\begin{cases} \tilde{X}_t = X_0 + \int_0^t f(s, \tilde{X}_s, Y_s, Z_s) ds + \int_0^t \sigma(s, \tilde{X}_{s-}, Y_{s-}) dH_s, \\ \tilde{Y}_t = \varphi(X_T) + \int_t^T g(s, \tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dH_s. \end{cases} \quad (4.1)$$

Lemma 4.1. *Assume that all the conditions in Theorem (3.1) are satisfied. Let $(\tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s)$ and $(\tilde{U}_t, \tilde{V}_t, \tilde{W}_s)$ belong to $M^2(0, T)$ and satisfy the equation (4.1), then there exists three constants c, c' and c'' depending on λ and λ_0 , such that the following estimates hold true*

$$\left(1 - cT^{\frac{1}{2}}\right) \mathbb{E} \sup_{0 \leq t \leq T} \left| \tilde{X}_t - \tilde{U}_t \right|^2 \leq cT^{\frac{1}{2}} \left(\mathbb{E} \sup_{0 \leq s \leq T} |Y_s - V_s|^2 + \mathbb{E} \int_0^T \|Z_s - W_s\|_{l^2(\mathbb{R})}^2 ds \right), \quad (4.2)$$

$$(1 - c''T) \mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \tilde{Y}_t - \tilde{V}_t \right|^2 \right) \leq c''(1 + T) \mathbb{E} \left(\sup_{0 \leq s \leq T} \left| \tilde{X}_s - \tilde{U}_s \right|^2 \right), \quad (4.3)$$

$$\mathbb{E} \left[\int_0^T \left\| \tilde{Z}_s - \tilde{W}_s \right\|_{l^2(\mathbb{R})}^2 ds \right] \leq c' \left((1 + T) \mathbb{E} \left(\sup_{0 \leq s \leq T} \left| \tilde{X}_s - \tilde{U}_s \right|^2 \right) + T \mathbb{E} \left(\sup_{0 \leq s \leq T} \left| \tilde{Y}_s - \tilde{V}_s \right|^2 \right) \right). \quad (4.4)$$

Proof of Lemma 4.1. Let us consider $(X_t, Y_t, Z_t)_{0 \leq t \leq T}$, $(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t)_{0 \leq t \leq T}$,

$(U_t, V_t, W_t)_{0 \leq t \leq T}$, $(\tilde{U}_t, \tilde{V}_t, \tilde{W}_t)_{0 \leq t \leq T} \in M^2(0, T)$. First, we proceed to prove (4.2). Applying Itô's formula to $\left| \tilde{X}_t - \tilde{U}_t \right|^2$, taking expectation and using the fact that $[H^i, H^j]_t - \langle H^i, H^j \rangle_t$ is an \mathcal{F}_t -martingale and $\langle H^i, H^j \rangle_t = \delta_{ij}t$, then there exists a constant c , depending on λ such that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \left| \tilde{X}_t - \tilde{U}_t \right|^2 &\leq c \left[\mathbb{E} \int_0^T \left| \tilde{X}_s - \tilde{U}_s \right| \left(\left| \tilde{X}_s - \tilde{U}_s \right| + |Y_s - V_s| + \|Z_s - W_s\|_{l^2(\mathbb{R})}^2 \right) ds \right. \\ &\quad \left. + \mathbb{E} \int_0^T \left(\left| \tilde{X}_s - \tilde{U}_s \right|^2 + |Y_s - V_s|^2 \right) ds \right] \\ &\quad + 2 \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \left(\tilde{X}_{s-} - \tilde{U}_{s-} \right) \sigma \left(s, \tilde{X}_{s-}, Y_{s-} \right) - \sigma \left(s, \tilde{U}_{s-}, V_{s-} \right) dH_s \right|. \end{aligned}$$

Burkholder-Davis-Gundy's inequality applied to the martingale

$$\int_0^t \left(\tilde{X}_s - \tilde{U}_s \right) \sigma \left(s, \tilde{X}_{s-}, Y_{s-} \right) - \sigma \left(s, \tilde{U}_{s-}, V_{s-} \right) dH_s$$

yields the existence of a constant $C > 0$, such that

$$\begin{aligned} &\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \left(\tilde{X}_s - \tilde{U}_s \right) \left(\sigma \left(s, \tilde{X}_{s-}, Y_{s-} \right) - \sigma \left(s, \tilde{U}_{s-}, V_{s-} \right) \right) dH_s \right| \\ &\leq C \mathbb{E} \left(\left[\int_0^t \left(\tilde{X}_{s-} - \tilde{U}_{s-} \right) \left(\sigma \left(s, \tilde{X}_{s-}, Y_{s-} \right) - \sigma \left(s, \tilde{U}_{s-}, V_{s-} \right) \right) dH_s \right]^{\frac{1}{2}} \right) \end{aligned}$$

Moreover, since $\langle H^i, H^i \rangle = \delta_{ii}t$ and $[M]_t = \langle M \rangle_t + \psi_t$, where ψ_t is a uniformly integrable

martingale starting at 0, then

$$\begin{aligned}
& \mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^t (\tilde{X}_s - \tilde{U}_s) \left(\sigma(s, \tilde{X}_{s-}, Y_{s-}) - \sigma(s, \tilde{U}_{s-}, V_{s-}) \right) dH_s \right|^2 \right)^{1/2} \\
& \quad C \mathbb{E} \left(\left[\int_0^t (\tilde{X}_s - \tilde{U}_s) \left(\sigma(s, \tilde{X}_{s-}, Y_{s-}) - \sigma(s, \tilde{U}_{s-}, V_{s-}) \right) dH_s \right]^2 \right)^{1/2} \\
& = C \mathbb{E} \left(\left\langle \int_0^t (\tilde{X}_{s-} - \tilde{U}_{s-}) \sigma(s, \tilde{X}_{s-}, Y_{s-}) - \sigma(s, \tilde{U}_{s-}, V_{s-}) dH_s \right\rangle + \psi_t \right)^{1/2} \\
& = C \mathbb{E} \left(\int_0^T \left| \tilde{X}_s - \tilde{U}_s \right|^2 \left\| \sigma(s, \tilde{X}_s, Y_s) - \sigma(s, \tilde{U}_s, V_s) \right\|_{l^2(\mathbb{R})}^2 ds \right)^{1/2}.
\end{aligned}$$

Then, modifying c if necessary, we have

$$\begin{aligned}
& \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \tilde{X}_t - \tilde{U}_t \right|^2 \right] \\
& \leq cT^{1/2} \left(\mathbb{E} \left(\sup_{0 \leq s \leq T} \left| \tilde{X}_s - \tilde{U}_s \right|^2 \right) + \mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_s - V_s|^2 \right) + \mathbb{E} \left(\int_0^T \|Z_s - W_s\|_{l^2(\mathbb{R})}^2 ds \right) \right);
\end{aligned}$$

which implies that,

$$(1 - cT^{1/2}) \mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \tilde{X}_t - \tilde{U}_t \right|^2 \right) \leq cT^{1/2} \left(\mathbb{E} \left(\sup_{0 \leq s \leq T} |Y_s - V_s|^2 \right) + \mathbb{E} \left(\int_0^T \|Z_s - W_s\|_{l^2(\mathbb{R})}^2 ds \right) \right).$$

On the other hand, by applying Itô's formula to $\left| \tilde{Y}_t - \tilde{V}_t \right|^2$, we get

$$\begin{aligned}
& \left| \tilde{Y}_t - \tilde{V}_t \right|^2 + \int_t^T \left\| \tilde{Z}_s - \tilde{W}_s \right\|_{l^2(\mathbb{R})}^2 ds \\
& = \left| \varphi(\tilde{X}_T) - \varphi(\tilde{U}_T) \right|^2 + 2 \int_t^T (\tilde{Y}_s - \tilde{V}_s) \left(g(s, \tilde{X}_s, \tilde{Y}_s, \tilde{Z}_s) - g(s, \tilde{U}_s, \tilde{V}_s, \tilde{W}_s) \right) ds \\
& \quad - 2 \int_t^T (\tilde{Y}_s - \tilde{V}_s) \left(\tilde{Z}_s - \tilde{W}_s \right) dH_s - \sum_{i,j} \int_t^T \left(\tilde{Z}_s^i - \tilde{W}_s^i \right) \left(\tilde{Z}_s^j - \tilde{W}_s^j \right) d[H^i, H^j]_s.
\end{aligned} \tag{4.5}$$

Thus, by taking expectations, invoking the assumption (\mathbf{H}_1) and using the fact that $(\tilde{Z}_s^i - \tilde{W}_s^i) - \langle H^i, H^j \rangle_t$ is an \mathcal{F}_t -martingale and $\langle H^i, H^j \rangle_t = \delta_{ij}t$, one can show that there exists a constant c' , depending on λ and λ_0 , such that

$$\begin{aligned}
& \mathbb{E} \int_0^T \left\| \tilde{Z}_s - \tilde{W}_s \right\|_{l^2(\mathbb{R})}^2 ds \leq c' \left[\mathbb{E} \left| \tilde{X}_T - \tilde{U}_T \right|^2 \right. \\
& \quad \left. + \mathbb{E} \int_0^T \left| \tilde{Y}_s - \tilde{V}_s \right| \left(\left| \tilde{X}_s - \tilde{U}_s \right| + \left| \tilde{Y}_s - \tilde{V}_s \right| + \left\| \tilde{Z}_s - \tilde{W}_s \right\|_{l^2(\mathbb{R})} \right) ds \right].
\end{aligned}$$

Using the fact that $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$ for any $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
& \mathbb{E} \int_0^T \left| \tilde{Z}_s - \tilde{W}_s \right|^2 ds \leq c' \left[(1 + T) \mathbb{E} \sup_{0 \leq s \leq T} \left| \tilde{X}_s - \tilde{U}_s \right|^2 \right. \\
& \quad \left. + T \mathbb{E} \sup_{0 \leq s \leq T} \left| \tilde{Y}_s - \tilde{V}_s \right|^2 \right] + \frac{1}{2} \mathbb{E} \int_0^T \left\| \tilde{Z}_s - \tilde{W}_s \right\|_{l^2(\mathbb{R})}^2 ds.
\end{aligned}$$

By modifying c' if necessary, we obtain

$$\begin{aligned} & \mathbb{E} \int_0^T \left\| \tilde{Z}_s - \tilde{W}_s \right\|_{l^2(\mathbb{R})}^2 ds \\ & \leq c' \left[(1+T) \mathbb{E} \sup_{0 \leq s \leq T} \left| \tilde{X}_s - \tilde{U}_s \right|^2 + T \mathbb{E} \sup_{0 \leq s \leq T} \left| \tilde{Y}_s - \tilde{V}_s \right|^2 \right]. \end{aligned} \quad (4.6)$$

Using equality (4.5) once again, and the Burkholder-Davis-Gundy inequality, we show that there exists a constant c'' , only depending on λ and λ_0 , such that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \left| \tilde{Y}_t - \tilde{V}_t \right|^2 & \leq c'' \left[\mathbb{E} \left| \tilde{X}_T - \tilde{U}_T \right|^2 + \mathbb{E} \left(\int_0^T \left| \tilde{Y}_s - \tilde{V}_s \right|^2 \left\| \tilde{Z}_s - \tilde{W}_s \right\|_{l^2(\mathbb{R})}^2 ds \right)^{1/2} \right. \\ & \quad \left. + \mathbb{E} \int_0^T \left| \tilde{Y}_s - \tilde{V}_s \right| \left(\left| \tilde{X}_s - \tilde{U}_s \right| + \left| \tilde{Y}_s - \tilde{V}_s \right| + \left\| \tilde{Z}_s - \tilde{W}_s \right\|_{l^2(\mathbb{R})}^2 \right) ds \right]. \end{aligned}$$

Then, Taking into account (4.6), using Young's inequality one more time, and modifying c'' if necessary, we get

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \tilde{Y}_t - \tilde{V}_t \right|^2 \right) & \leq c'' \left[(1+T) \mathbb{E} \left(\sup_{0 \leq s \leq T} \left| \tilde{X}_s - \tilde{U}_s \right|^2 \right) + T \mathbb{E} \left(\sup_{0 \leq s \leq T} \left| \tilde{Y}_s - \tilde{V}_s \right|^2 \right) \right] \\ & + \frac{1}{2} \mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \tilde{Y}_t - \tilde{V}_t \right|^2 \right) \end{aligned}$$

Then, modifying c'' if necessary, we have

$$(1 - c''T) \mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \tilde{Y}_t - \tilde{V}_t \right|^2 \right) \leq c'' (1+T) \mathbb{E} \left(\sup_{0 \leq s \leq T} \left| \tilde{X}_s - \tilde{U}_s \right|^2 \right).$$

Lemma 4.1 is proved. \square

Proof of Theorem 3.1. Let $(X_t, Y_t, Z_t)_{0 \leq t \leq T}$ be a possible solution of FBSDE (1.1) and $(\tilde{X}, \tilde{Y}, \tilde{Z})$ be defined as in Lemma 4.1. It is clear that the process \tilde{X} is a solution of a Forward component of the SDE (4.1), whereas the couple (\tilde{X}, \tilde{Y}) is a solution of a Backward component of the SDE (4.1) SDE. Then $(\tilde{X}, \tilde{Y}, \tilde{Z})$ is a solution of the above decoupled Forward Backward SDE (4.1). To prove the existence and the uniqueness of the solution in $M^2(0, T)$, we use the fixed point method. Let us define a mapping Ψ from $M^2(0, T)$ into itself defined by

$$\Psi(X, Y, Z) = (\tilde{X}, \tilde{Y}, \tilde{Z}).$$

We want to prove that there exists a constant $\delta > 0$, only depending on λ and λ_0 , such that for $T \leq \delta$, Ψ is a contraction on $M^2(0, T)$ equipped with the norm

$$\|\Psi(X, Y, Z)\|_{M^2(0, T)}^2 = \mathbb{E} \left\{ \sup_{0 \leq t \leq T} [|X_t|^2 + |Y_t|^2] + \int_0^T \|Z_t\|_{l^2(\mathbb{R})}^2 dt \right\}.$$

In order to achieve this goal, we firstly assume that $T \leq 1$. Further, we set

$$\Psi(X, Y, Z) = (\tilde{X}, \tilde{Y}, \tilde{Z}), \quad \Psi(U, V, W) = (\tilde{U}, \tilde{V}, \tilde{W}).$$

where $(X_t, Y_t, Z_t)_{0 \leq t \leq T}, (U_t, V_t, W_t)_{0 \leq t \leq T}$ be two elements of $M^2(0, T)$. Thus, by invoking and combining the results (4.2), (4.3) and (4.4) of the Lemma 4.1, a simple computation shows that there exists a constant δ depending on λ and λ_0 , such that for $T \leq \delta$, the following estimate holds true

$$\|\Psi(X, Y, Z) - \Psi(U, V, W)\|_{M^2(0, T)} \leq D \|(X, Y, Z) - (U, V, W)\|_{M^2(0, T)},$$

For some constant $0 < D < 1$. This proves that the map Ψ is contraction from $M^2(0, T)$ into itself. Furthermore, It follows immediately that this mapping has a unique fixed point (X_t, Y_t, Z_t) progressively measurable which is the unique solution of FBSDE (1.1). The proof is complete. \square

Proof of Proposition 3.1. Arguing as in the proof of Lemma 4.1 and standard arguments of FBSDEs (see for example [1] for the Brownian case), one can prove (i). Now we proceed to prove (ii). For this end, let us define the stopping time

$$R_k = \inf \{t : |X_t| > k\} \text{ with } X_0 = 0.$$

For each k , denoting $(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X1_{[0, R_k]}, Y1_{[0, R_k]}, Z1_{[0, R_k]})$. It is clear that the stopped process $\tilde{X} = X1_{[0, R_k]}$ is bounded by k , and is a semimartingale as a product of two semimartingales, which is valid for \tilde{Y} as well. Therefore, by applying Itô's formula, using the fact that $[H^i, H^j]_t - \langle H^i, H^j \rangle_t$ is an \mathcal{F}_t -martingale, $\langle H^i, H^j \rangle_t = \delta_{ij}t$ and standart techniques from FBSDE theory, one can prove that

$$\begin{aligned} & \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \left[|\tilde{X}_t|^{2p} + |\tilde{Y}_t|^{2p} \right] + \left(\int_0^T \|\tilde{Z}_t\|_{l^2(\mathbb{R})}^2 dt \right)^p \right\} \\ & \leq C_1 \mathbb{E} \left\{ |\tilde{X}_0|^{2p} + |\tilde{\varphi}(0)|^{2p} + \int_0^T \left[|\tilde{f}(t, 0, 0, 0)|^{2p} + \|\tilde{\sigma}(t, 0, 0)\|_{l^2(\mathbb{R})}^{2p} + |\tilde{g}(t, 0, 0, 0)|^{2p} \right] dt \right\} \\ & + \sum_{0 < s \leq t} \left\{ \left(\tilde{X}_s^{2p} \right) - \tilde{X}_{s-}^{2p} - 2p \tilde{X}_s^{2p-1} \Delta \tilde{X}_s - p(2p-1) \tilde{X}_s^{2p-2} \left(\Delta \tilde{X}_s \right)^2 \right\} \\ & + \sum_{t < s \leq T} \left\{ \tilde{Y}_s^{2p} - \tilde{Y}_{s-}^{2p} - 2p \tilde{Y}_s^{2p-1} \Delta \tilde{Y}_s - p(2p-1) \tilde{Y}_s^{2p-2} \left(\Delta \tilde{Y}_s \right)^2 \right\}, \end{aligned} \quad (4.7)$$

where we have denoted by $\tilde{\varphi}, \tilde{f}, \tilde{\sigma}$, and \tilde{g} the restriction of the functions of φ, f, σ , and g . Now, we proceed to prove that

$$\sum_{0 < s \leq t} \left\{ X_s^{2p} - X_{s-}^{2p} - 2p X_s^{2p-1} \Delta X_s - p(2p-1) X_s^{2p-2} (\Delta X_s)^2 \right\} < C [X, X]_t.$$

Since \tilde{X} takes its values in intervals of the form $[-k, k]$, for $h(x) = x^{2p}$, it is easy to show that

$$\left| h(x) - h(y) - (y-x) h'(x) - (y-x)^2 h''(x) \right| \leq C (y-x)^2$$

Thus

$$\begin{aligned} & \sum_{0 < s \leq t} \left| \tilde{X}_s^{2p} - \tilde{X}_{s-}^{2p} - 2p \tilde{X}_s^{2p-1} \Delta \tilde{X}_s - p(2p-1) \tilde{X}_s^{2p-2} \left(\Delta \tilde{X}_s \right)^2 \right| \\ & \leq C \sum_{0 < s \leq t}^2 \left(\Delta \tilde{X}_s \right) \leq C [\tilde{X}, \tilde{X}]_t < \infty. \end{aligned} \quad (4.8)$$

Therefore by similar arguments developed above, one can easily derive that

$$\begin{aligned} & \sum_{t < s \leq T} \left| \tilde{Y}_s^{2p} - \tilde{Y}_{s-}^{2p} - 2p\tilde{Y}_s^{2p-1}\Delta\tilde{Y}_s - p(2p-1)\tilde{Y}_s^{2p-2}(\Delta\tilde{Y}_s)^2 \right| \\ & \leq C \sum_{t < s \leq T} (\Delta\tilde{Y}_s)^2 \leq C \left([\tilde{Y}, \tilde{Y}]_T - [\tilde{Y}, \tilde{Y}]_t \right) < \infty. \end{aligned} \quad (4.9)$$

Combining (4.8), (4.9) and (4.7), we get

$$\begin{aligned} & \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \left[|\tilde{X}_t|^{2p} + |\tilde{Y}_t|^{2p} \right] + \left(\int_0^T \|\tilde{Z}_t\|_{l^2(\mathbb{R})}^2 dt \right)^p \right\} \\ & \leq C_1 \mathbb{E} \left\{ |\tilde{X}_0|^{2p} + |\varphi(0)|^{2p} + \int_0^T \left[|f(t, 0, 0, 0)|^{2p} + \|\tilde{\sigma}(t, 0, 0)\|_{l^2(\mathbb{R})}^{2p} + |g(t, 0, 0, 0)|^{2p} \right] dt \right\} + C < \infty. \end{aligned}$$

Since the last inequality is valid for $(\tilde{X}, \tilde{Y}, \tilde{Z})$ for each k , it also remains valid for (X, Y, Z) and this completes the proof. \square

4.2 Large time duration

To prove Theorem 3.2, we need the following proposition, which allows us to prove global existence and uniqueness of the equation (1.1). By using similar arguments introduced in [16] consisting in solving the system iteratively in small intervals having fixed length.

Proposition 4.1. *Let $\Pi^i, i = 0, 1$, be the solution to FBSDEs:*

$$\begin{cases} X_t^i = x_i + \int_0^t f(s, \Pi_s^i) ds + \int_0^t \sigma(s, X_{s-}^i, Y_{s-}^i) dH_s, \\ Y_t^i = \varphi(X_T^i) + \int_t^T g(s, \Pi_s^i) ds - \int_t^T Z_s^i dH_s. \end{cases}$$

Assume that (\mathbf{H}_1) is satisfied and $V_0^2 < \infty$. Then

$$|Y_0^1 - Y_0^0| \leq \bar{\lambda}_0 |x_1 - x_0|,$$

where

$$\bar{\lambda}_0 = c \left([\lambda_0 + 1] e^{(2\lambda + \lambda^2)T} - 1 \right). \quad (4.10)$$

The following lemma gives estimates of $\bar{\lambda}_0$ in terms of λ and λ_0 . This estimation is the key step for the proof of Theorem 3.2.

Lemma 4.2. *Consider the following linear FBSDE:*

$$\begin{cases} X_t = 1 + \int_0^t (a_s^1 X_s + b_s^1 Y_s + c_s^1 Z_s) ds + \int_0^t (a_s^2 X_s + b_s^2 Y_s) dH_s, \\ Y_t = FX_T + \int_t^T (a_s^3 X_s + b_s^3 Y_s + c_s^3 Z_s) ds - \int_t^T Z_s dH_s. \end{cases} \quad (4.11)$$

Assume $|a_t^i|, |b_t^i|, |c_t^i| \leq \lambda, i = 1, 2, 3$ and $|F| \leq \lambda_0$. Let δ be as in theorem 3.1. And assume further that

$$b_t^2 c_t^1 = 0; \quad b_t^1 + a_t^2 c_t^1 + b_t^2 c_t^3 = 0. \quad (4.12)$$

Then for $T \leq \delta$,

- i) The LFBSDE (4.11) admits a unique solution.
- ii)

$$|Y_0| \leq \bar{\lambda}_0, \quad (4.13)$$

where $\bar{\lambda}_0$ is defined by (4.10).

Proof of Lemma 4.2. First, we can easily check that LFBSDE (4.11) satisfy assumptions of Theorem 3.1, then it has a unique solution (X_t, Y_t, Z_t) which belongs to the space $M^2(0, T)$. This gives the proof of the assertion (i).

We shall prove the assertion (ii). We split the proof into two steps.

Step1. For any $t \in [0, T)$ and any $\xi \in L^2(\mathcal{F}_0)$, we put $\bar{\Pi}_s \triangleq (X_t \xi, Y_t \xi, Z_t \xi)$, $s \in [t, T]$. Then $\bar{\Pi}_s$ satisfies the following linear FBSDE

$$\begin{cases} \bar{X}_s = X_t \xi + \int_t^s [a_r^1 \bar{X}_r + b_r^1 \bar{Y}_r + c_r^1 \bar{Z}_r] dr + \int_t^s [a_r^2 \bar{X}_r + b_r^2 \bar{Y}_r] dH_r, \\ \bar{Y}_s = F \bar{X}_T + \int_s^T [a_r^3 \bar{X}_r + b_r^3 \bar{Y}_r + c_r^3 \bar{Z}_r] dr - \int_s^T \bar{Z}_r dH_r. \end{cases}$$

By assertion (ii) of Theorem 3.1, we get

$$E \left\{ |\bar{Y}_t|^2 \right\} = E \left\{ |Y_t \xi|^2 \right\} \leq C_0^2 E \left\{ |X_t \xi|^2 \right\}.$$

Since ξ is arbitrary, we have $|Y_t| \leq C_0 |X_t|$, P -a.s., $\forall t$.

Step2. We define

$$\tau \triangleq \inf \{t > 0 : X_t = 0\} \wedge T; \quad \text{and} \quad \tau_n \triangleq \inf \left\{ t > 0 : X_t = \frac{1}{n} \right\} \wedge T.$$

Then $\tau_n \uparrow \tau$ and $X_t > 0$ for $t \in [0, \tau)$. We also define the pure jump process η , by the following formula

$$\eta_t = \prod_{0 < s \leq t} (1 - (X_s)^{-1} \Delta X_s) \frac{(X_{s-})^{-1}}{(X_s)^{-1}}$$

The above product is clearly càdlàg, adapted, converges and is of finite variation. We put for any $t \in [0, \tau)$,

$$A_t = \eta_t (X_t)^{-1}.$$

It should be noted that when we apply Itô's formula to $(X_t)^{-1}$, a sum of discontinuous quantities appears. To eliminate this, we shall apply Itô's formula to $A_t = \eta_t (X_t)^{-1}$ instead of $(X_t)^{-1}$. Firstly, applying Itô's formula to A_t , we have

$$\begin{aligned} A_t = & A_0 - \int_0^t \eta_{s-} (X_{s-})^{-2} dX_s + \int_0^t (X_{s-})^{-1} d\eta_s + \int_0^t A_{s-} (X_s)^{-2} d[X, X]_s^c \\ & + \sum_{0 < s \leq t} (A_s - A_{s-} + A_{s-} (X_{s-})^{-1} (\Delta X_s) - (X_{s-})^{-1} \Delta \eta_s), \end{aligned} \quad (4.14)$$

Note that η is a pure jump process. Hence $[\eta, X]^c = [\eta, \eta]^c = 0$ and

$$\int_0^t \left(\tilde{X}_{s-} \right)^{-1} d\eta_s = \sum_{0 < s \leq t} \left(\tilde{X}_{s-} \right)^{-1} \Delta \eta_s.$$

Then (4.14) becomes

$$\begin{aligned} A_t &= A_0 - \int_0^t \eta_{s-} (X_{s-})^{-2} dX_s + \int_0^t A_{s-} (X_{s-})^{-2} d[X, X]_s^c \\ &\quad + \sum_{0 < s \leq t} (A_s - A_{s-} + A_{s-} (X_{s-})^{-1} \Delta X_s), \end{aligned}$$

The following equality is obvious, from the definition of the process A ,

$$A_s = A_{s-} (1 - (X_t)^{-1} \Delta X_t).$$

Now by replacing the above equality into the previous one, one can get

$$\sum_{0 < s \leq t} (A_s - A_{s-} + A_{s-} (X_{s-})^{-1} \Delta X_s) = 0.$$

Therefore,

$$A_t = A_0 - \int_0^t A_s (X_s)^{-1} dX_s + \int_0^t A_{s-} (X_s)^{-2} d[X, X]_s^c,$$

with

$$d[X, X]_s^c = \sum_{i,j} (a_s^{2,i} X_s + b_s^{2,i} Y_s) (a_s^{2,j} X_s + b_s^{2,j} Y_s) q_{i-1}(0) q_{j-1}(0) ds.$$

Thanks to Lemma 2.1, we get

$$\begin{aligned} d[X, X]_s^c &= \left[(a_s^2 X_s + b_s^2 Y_s)^2 - \sum_{i,j} (a_s^{2,i} X_s + b_s^{2,i} Y_s) (a_s^{2,j} X_s + b_s^{2,j} Y_s) \int_{\mathbb{R}} p_i(x) p_j(x) v(dx) \right] ds \\ &= \left[(a_s^2 X_s + b_s^2 Y_s)^2 - \Psi_s \right] ds \end{aligned}$$

Hence

$$\begin{aligned} A_t &= A_0 - \int_0^t \left[A_s (X_s)^{-1} (a_s^1 X_s + b_s^1 Y_s + c_s^1 Z_s) - A_s (X_s)^{-2} (a_s^2 X_s + b_s^2 Y_s)^2 \right] ds \\ &\quad - \int_0^t A_{s-} (X_{s-})^{-1} (a_s^2 X_{s-} + b_s^2 Y_{s-}) dH_s - \int_0^t A_s (X_s)^{-2} \Psi_s ds. \end{aligned}$$

Let us define the following processes

$$\hat{Y}_t = Y_t A_t; \quad \hat{Z}_t \triangleq A_t Z_t - A_t (X_t)^{-1} Y_t (a_t^2 X_t + b_t^2 Y_t).$$

Then after the result of the Step 1, we have

$$|\hat{Y}_t| \leq C_0.$$

Now, applying Itô's formula to \hat{Y}_t , we obtain

$$\begin{aligned} d\hat{Y}_t &= -A_t (a_t^3 X_t + b_t^3 Y_t + c_t^3 Z_t) dt \\ &\quad - \left[Y_t A_t (X_t)^{-1} (a_t^1 X_t + b_t^1 Y_t + c_t^1 Z_t) dt - A_t (X_t)^{-2} (a_t^2 X_t + b_t^2 Y_t)^2 \right] dt \\ &\quad - \left[A_t (X_t)^{-1} (a_t^2 X_t + b_t^2 Y_t) Z_t \right] dt - [Y_t A_t (X_t)^{-2} \Psi_t] dt \\ &\quad + [A_{t-} Z_t - Y_{t-} A_{t-} (X_{t-})^{-1} (a_t^2 X_{t-} + b_t^2 Y_{t-})] dH_t + d\tilde{A}_t. \end{aligned}$$

where we have denoted by $\tilde{A}_t = [A, Y]_t - \langle A, Y \rangle_t$. By using the definition of the processes (\hat{Y}, \hat{Z}) it follows that

$$\begin{aligned} d\hat{Y}_t &= \hat{Z}_t dH_t - \left[c_t^3 + c_t^1 \eta_t^{-1} \hat{Y}_t + a_t^2 + b_t^2 \eta_t^{-1} \hat{Y}_t \right] \hat{Z}_t dt \\ &- \left[c_t^1 b_t^2 (\eta_t^{-1})^2 \hat{Y}_t^3 + (b_t^1 + a_t^2 c_t^1 + c_t^3 b_t^2) \eta_t^{-1} \hat{Y}_t^2 \right] dt - \left[a_t^3 \eta_t + (b_t^3 + a_t^1 + c_t^3 a_t^2) \hat{Y}_t \right] dt \\ &- \left[Y_t A_t (X_t)^{-2} \Psi_t \right] dt + d\tilde{A}_t. \end{aligned}$$

Thus, by taking into account (4.12),

$$\begin{aligned} d\hat{Y}_t &= \hat{Z}_t dH_t - \left[c_t^3 + c_t^1 \eta_t^{-1} \hat{Y}_t + a_t^2 + b_t^2 \eta_t^{-1} \hat{Y}_t \right] \hat{Z}_t dt \\ &- \left[a_t^3 \eta_t + (b_t^3 + a_t^1 + c_t^3 a_t^2) \hat{Y}_t \right] dt - \left[Y_t A_t (X_t)^{-2} \Psi_t \right] dt + d\tilde{A}_t. \end{aligned}$$

We put

$$\begin{aligned} \Gamma_t &= 1 + \int_0^t \Gamma_s (X_s)^{-2} \Psi_s 1_{\{\tau > s\}} ds. \\ M_t &= 1 + \sum_{i=1}^{\infty} \int_0^t (q_{i-1}(0))^{-1} M_s \left((c_s^3 + a_s^2) + (c_s^1 + b_s^2) \eta_s^{-1} \hat{Y}_s \right) 1_{\{\tau > s\}} dB_s; \\ N_t &= 1 + \int_0^t N_s (a_s^1 + b_s^3 + a_s^2 c_s^3) 1_{\{\tau > s\}} ds. \end{aligned}$$

Applying Itô's formula to $(\Gamma_t N_t M_t \hat{Y}_t)$, we obtain

$$\begin{aligned} d(\Gamma_t N_t M_t \hat{Y}_t) &= \Gamma_t N_t M_t \hat{Z}_t 1_{\{\tau > t\}} dH_t + \sum_{i=1}^{\infty} \left(\Gamma_t \hat{Y}_t N_t M_t (q_{i-1}(0))^{-1} (c_t^3 + a_t^2) + (c_t^1 + b_t^2) \eta_t^{-1} \hat{Y}_t \right) 1_{\{\tau > t\}} dB_t \\ &- \eta_t \Gamma_t N_t M_t a_t^3 1_{\{\tau > s\}} dt + \Gamma_t M_t N_t d\tilde{A}_t. \end{aligned}$$

Taking expectations, we get

$$Y_0 = E \left(\Gamma_{\tau_n} N_{\tau_n} M_{\tau_n} \hat{Y}_{\tau_n} + \int_0^{\tau_n} \eta_t \Gamma_t N_t M_t a_t^3 dt \right). \quad (4.15)$$

Since $|\hat{Y}_t| \leq C_0$, M is an \mathcal{F}_t -martingale and $|N_t| \leq e^{(2\lambda + \lambda^2)t}$. Moreover, we observe that

if $\tau = T$, $|Y_\tau| = |Y_T| = |FX_T| = |FX_\tau| \leq \lambda_0 |X_\tau|$.

If $\tau < T$, $X_\tau = 0$, and thus $|Y_\tau| \leq C_0 |X_\tau| = 0$.

Therefore, in both cases it holds that $|Y_\tau| \leq \lambda_0 |X_\tau|$.

Now, applying Ito's formula to $|Y_t|^2$ from $s = \tau_n$ to $s = \tau$, we obtain

$$\begin{aligned} |Y_{\tau_n}|^2 + E_{\tau_n} \left(\int_{\tau_n}^{\tau} \|Z_t\|_{l^2(\mathbb{R})}^2 dt \right) &= E_{\tau_n} \left(|Y_\tau|^2 + 2 \int_{\tau_n}^{\tau} Y_t (a_t^3 X_t + b_t^3 Y_t + c_t^3 Z_t) dt \right) \\ &\leq E_{\tau_n} \left(\lambda_0^2 |X_\tau|^2 + C \int_{\tau_n}^{\tau} (|X_t|^2 + |Y_t|^2) dt + \frac{1}{2} \int_{\tau_n}^{\tau} \|Z_t\|_{l^2(\mathbb{R})}^2 dt \right). \end{aligned}$$

Similarly, applying Ito's formula to $|X_t|^2$ from $s = \tau_n$ to $s = \tau$, we obtain,

$$E_{\tau_n} (|X_\tau|^2) \leq E_{\tau_n} \left(|X_{\tau_n}|^2 + C \int_{\tau_n}^{\tau} (|X_t|^2 + |Y_t|^2) dt + \frac{1}{2\lambda_0^2} \int_{\tau_n}^{\tau} \|Z_t\|_{l^2(\mathbb{R})}^2 dt \right).$$

Thus

$$|Y_{\tau_n}|^2 \leq E_{\tau_n} \left(\lambda_0^2 |X_{\tau_n}|^2 + C \int_{\tau_n}^{\tau} (|X_t|^2 + |Y_t|^2) dt \right).$$

Note that $|X_{\tau_n}| \geq \frac{1}{n}$, then

$$\begin{aligned} |\hat{Y}_{\tau_n}| &\leq \lambda_0 |\eta_{\tau_n}| + C E_{\tau_n}^{\frac{1}{2}} \left(\int_{\tau_n}^{\tau} (|\tilde{X}_t|^2 + |\tilde{Y}_t|^2) dt \right) \\ &\leq \lambda_0 |\eta_{\tau_n}| + C E_{\tau_n}^{\frac{1}{2}} \left(\sup_{\tau_n \leq t \leq \tau} (|\tilde{X}_t|^2 + |\tilde{Y}_t|^2) (\tau - \tau_n) \right), \end{aligned}$$

where

$$\tilde{X}_t \triangleq X_t |\eta_{\tau_n}| (X_{\tau_n})^{-1} ; \quad \tilde{Y}_t \triangleq Y_t |\eta_{\tau_n}| (X_{\tau_n})^{-1}.$$

Now by (4.15), we get

$$\begin{aligned} |\hat{Y}_0| &\leq \lambda E(\Gamma_t M_t) \int_0^T |\eta_t| e^{(2\lambda+\lambda^2)t} dt \\ &+ E \left\{ e^{(2\lambda+\lambda^2)T} M_{\tau_n} \Gamma_{\tau_n} \left(|\eta_{\tau_n}| \lambda_0 + C E_{\tau_n}^{\frac{1}{2}} \left(\sup_{\tau_n \leq t \leq \tau} (|\tilde{X}_t|^2 + |\tilde{Y}_t|^2) (\tau - \tau_n) \right) \right) \right\} \\ &\leq c' \left(e^{(2\lambda+\lambda^2)T} - 1 \right) + c'' \lambda_0 e^{(2\lambda+\lambda^2)T} \\ &+ C E \left\{ M_{\tau_n} \Gamma_{\tau_n} E_{\tau_n}^{\frac{1}{2}} \left(\sup_{\tau_n \leq t \leq \tau} (|\tilde{X}_t|^2 + |\tilde{Y}_t|^2) (\tau - \tau_n) \right) \right\} \\ &\leq \bar{\lambda}_0 + C E^{\frac{1}{2}} (|M_{\tau_n}|^2 |\Gamma_{\tau_n}|^2) E^{\frac{1}{2}} \left(\sup_{\tau_n \leq t \leq \tau} (|\tilde{X}_t|^2 + |\tilde{Y}_t|^2) (\tau - \tau_n) \right) \\ &\leq \bar{\lambda}_0 + C E^{\frac{1}{4}} \left(\sup_{\tau_n \leq t \leq \tau} (|\tilde{X}_t|^4 + |\tilde{Y}_t|^4) \right) E^{\frac{1}{4}} (|\tau - \tau_n|^2). \end{aligned}$$

Note that $(\tilde{X}_t, \tilde{Y}_t)$ satisfies the following LFBSDE:

$$\begin{cases} \tilde{X}_t = 1 + \int_0^t \left[a_r^1 1_{\{\tau_n \leq r\}} \tilde{X}_r + b_r^1 1_{\{\tau_n \leq r\}} \tilde{Y}_r + c_r^1 1_{\{\tau_n \leq r\}} \tilde{Z}_r \right] dr \\ \quad + \int_0^t \left[a_r^2 1_{\{\tau_n \leq r\}} \tilde{X}_r + b_r^2 1_{\{\tau_n \leq r\}} \tilde{Y}_r \right] dH_r, \\ \tilde{Y}_t = F \tilde{X}_T + \int_t^T \left[a_r^3 1_{\{\tau_n \leq r\}} \tilde{X}_r + b_r^3 1_{\{\tau_n \leq r\}} \tilde{Y}_r + c_r^3 1_{\{\tau_n \leq r\}} \tilde{Z}_r \right] dr - \int_t^T \tilde{Z}_r dH_r. \end{cases}$$

By (ii) of Proposition 3.1, we have

$$E \left(\sup_{\tau_n \leq t \leq \tau} (|\tilde{X}_t|^4 + |\tilde{Y}_t|^4) \right) \leq E \left(\sup_{0 \leq t \leq T} (|\tilde{X}_t|^4 + |\tilde{Y}_t|^4) \right) \leq C_1.$$

Thus

$$|\hat{Y}_0| \leq \bar{\lambda}_0 + C E^{\frac{1}{4}} (|\tau - \tau_n|^2).$$

Then for $n \rightarrow \infty$, we get $|\hat{Y}_0| \leq \bar{\lambda}_0$. That is, $|Y_0| \leq \bar{\lambda}_0 |X_0| |\eta_0| = \bar{\lambda}_0$. This complete the proof. \square

Proof of Proposition 4.1. The proof is the same as in Corollary 1 in [16], by replacing the Brownian part by the Teugels martingales and using the above lemma. \square

Now we are able to give the proof of our main result. We shall extend by induction the theorem 3.1 to 3.2.

Proof of Theorem 3.2. First we prove (i). Let λ and λ_0 be as in Theorem 3.1, and $\bar{\lambda}_0$ is a constant defined as in (4.10). Let δ be a constant as in Theorem 3.1, but corresponding to λ and $\bar{\lambda}_0$ instead of λ and λ_0 . For some integer n , we assume $(n-1)\delta < T \leq n\delta$ and consider a partition of $[0, T]$, with $T_i \triangleq \frac{iT}{n}, i = 0, \dots, n$.

We consider the mapping:

$$\begin{aligned} G_n : \quad \Omega \times \mathbb{R} &\rightarrow \mathbb{R} \\ \omega \times x &\mapsto \varphi(\omega, x) \end{aligned}$$

Let us consider the following FBSDE over the small interval $[T_{n-1}, T_n]$,

$$\begin{cases} X_t^n = x + \int_{T_{n-1}}^t f(s, \Pi_s^n) ds + \int_{T_{n-1}}^t \sigma(s, X_{s-}^n, Y_{s-}^n) dH_s, \\ Y_t^n = G_n(X_{T_n}^n) + \int_t^{T_n} g(s, \Pi_s^n) ds - \int_t^{T_n} Z_s^n dH_s. \end{cases} \quad (4.16)$$

Let L_{G_n} denotes the Lipschitz constant of the mapping G_n . Then, by Theorem 3.1 the required solution of FBSDE (4.16) exists and is unique. Define $G_{n-1}(x) \triangleq Y_{T_{n-1}}^n$, then for fixed x , $G_{n-1}(x) \in \mathcal{F}_{T_{n-1}}$. Further, in view of the Proposition 4.1, it's straightforward to verify that

$$L_{G_{n-1}} \leq \lambda_1 \triangleq c \left([\lambda_0 + 1] e^{(2\lambda + \lambda^2)(T_n - T_{n-1})} - 1 \right) \leq \bar{\lambda}_0.$$

Next, for $t \in [T_{n-2}, T_{n-1}]$, we consider the following FBSDE:

$$\begin{cases} X_t^{n-1} = x + \int_{T_{n-2}}^t f(s, \Pi_s^{n-1}) ds + \int_{T_{n-2}}^t \sigma(s, X_{s-}^{n-1}, Y_{s-}^{n-1}) dH_s, \\ Y_t^{n-1} = G_{n-1}(X_{T_{n-1}}^{n-1}) + \int_t^{T_{n-1}} g(s, \Pi_s^{n-1}) ds - \int_t^{T_{n-1}} Z_s^{n-1} dH_s. \end{cases} \quad (4.17)$$

Once again, since $L_{G_{n-1}} \leq \bar{\lambda}_0$, by Theorem 3.1, the FBSDE (4.17) has a unique solution.

Then as well, we may define $G_{n-2}(x)$, such that

$$\begin{aligned} L_{G_{n-2}} &\leq \lambda_2 \triangleq c \left([\lambda_1 + 1] e^{(2\lambda + \lambda^2)(T_{n-1} - T_{n-2})} - 1 \right) \\ &= c \left([\lambda_0 + 1] e^{(2\lambda + \lambda^2)(T_n - T_{n-2})} - 1 \right) \leq \bar{\lambda}_0. \end{aligned}$$

Repeating this procedure backwardly for $i = n, \dots, 1$, we may define G_i such that

$$L_{G_i} \leq \lambda_{n-i} \triangleq c \left([\lambda_0 + 1] e^{(2\lambda + \lambda^2)(T_n - T_i)} - 1 \right) \leq \bar{\lambda}_0.$$

As a conclusion, one can repeat the above construction and, after a finite number of steps, we obtain the required unique solution in each subinterval of the type $[T_{n-i}, T_{n-i}]$ for $i = 0, \dots, n$.

Now, for $i = 1, 2, \dots, n$ and for any $X_0 \in L^2(\mathcal{F}_0)$, we construct a solution for the following FBSDE

$$\begin{cases} X_t = X_{T_{i-1}} + \int_{T_{i-1}}^t f(s, \Pi_s) ds + \int_{T_{i-1}}^t \sigma(s, X_{s-}, Y_{s-}) dH_s, \\ Y_t = G_i(X_{T_i}) + \int_t^{T_i} g(s, \Pi_s) ds - \int_t^{T_i} Z_s dH_s. \end{cases} \quad t \in [t_{i-1}, t_i]$$

Obviously this provides a solution to the FBSDE (1.1). From the construction and the uniqueness of each step, it is clear that this solution is unique.

Now, let us prove (ii). We denote

$$V_t^2 = |f(t, 0, 0, 0)|^2 + |\sigma(t, 0, 0)|^2 + |g(t, 0, 0, 0)|^2.$$

From Theorem 3.1 and by the definition of G_i , we get

$$E \{ |G_{i-1}(0)|^2 \} \leq C_0 E \left\{ |G_i(0)|^2 + \int_{T_{i-1}}^{T_i} V_t^2 dt \right\}.$$

By induction one can easily prove that

$$\begin{aligned} \max_{0 \leq i \leq n} E \{ |G_i(0)|^2 \} &\leq C_0^n E \left\{ |\varphi(0)|^2 + \int_0^T V_t^2 dt \right\} \\ &= CE \left\{ |\varphi(0)|^2 + \int_0^T V_t^2 dt \right\}. \end{aligned}$$

Set $n \leq \frac{T}{\delta} + 1$ is a fixed constant depending only on λ, λ_0 and T , then so is C . Now for $t \in [T_0, T_1]$, by using (ii) of Theorem 3.1, we get

$$\begin{aligned} E \left\{ \sup_{0 \leq t \leq T} |X_t|^2 + \sup_{0 \leq t \leq T} |Y_t|^2 \right\} &\leq CE \left\{ |X_0|^2 + |G_1(0)|^2 + \int_{T_0}^{T_1} V_t^2 dt \right\} \\ &\leq CE \left\{ |X_0|^2 + |\varphi(0)|^2 + \int_0^T V_t^2 dt \right\}. \end{aligned}$$

Then by induction one can prove

$$E \left\{ \sup_{0 \leq t \leq T} |X_t|^2 + \sup_{0 \leq t \leq T} |Y_t|^2 \right\} \leq CE \left\{ |X_0|^2 + |\varphi(0)|^2 + \int_0^T V_t^2 dt \right\}. \quad (4.18)$$

On the other hand, applying Ito's formula to Y_t , we obtain

$$\begin{aligned} E \left\{ |Y_0|^2 + \int_0^T |Z_t|^2 dt \right\} &= E \left\{ |Y_T|^2 + 2 \int_0^T Y_t g(t, \Pi_t) dt \right\} \\ &\leq E \left\{ |Y_T|^2 + C \int_0^T [|g(t, 0, 0, 0)|^2 + |X_t|^2 + |Y_t|^2] dt + \frac{1}{2} \int_0^T |Z_t|^2 dt \right\}. \end{aligned}$$

Therefore

$$E \left\{ \int_0^T |Z_t|^2 dt \right\} \leq CE \left\{ |X_0|^2 + |\varphi(0)|^2 + \int_0^T V_t^2 dt \right\}. \quad (4.19)$$

Finally, combining (4.18) and (4.19) leads to $\|\Pi\|^2 \leq CV_0^2$, which achieves the proof. \square

4.3 Proof of stability theorem

Proof of Theorem 3.3. For $0 \leq \varepsilon \leq 1$, let Π^ε be the solution to the following FBSDE

$$\begin{cases} X_t^\varepsilon = X_0 + \varepsilon \Delta X_0 + \int_0^t (f^0(s, \Pi_s^\varepsilon) + \varepsilon \Delta f(s, \Pi_s^1)) ds \\ \quad + \int_0^t (\sigma^0(s, X_{s-}^\varepsilon, Y_{s-}^\varepsilon) + \varepsilon \Delta \sigma(s, X_{s-}^1, Y_{s-}^1)) dH_s; \\ Y_t^\varepsilon = (\varphi^0(X_T^\varepsilon) + \varepsilon \Delta \varphi(X_T^1)) + \int_t^T (g^0(s, \Pi_s^\varepsilon) + \varepsilon \Delta g(s, \Pi_s^1)) ds - \int_t^T Z_s^\varepsilon dH_s. \end{cases}$$

and $\nabla \Pi^\varepsilon$ be the solution of the following variational linear FBSDE

$$\begin{cases} \nabla X_t^\varepsilon = \Delta X_0 + \int_0^t (f_x^0(s, \Pi_s^\varepsilon) \nabla X_s^\varepsilon + f_y^0(s, \Pi_s^\varepsilon) \nabla Y_s^\varepsilon + f_z^0(s, \Pi_s^\varepsilon) \nabla Z_s^\varepsilon + \Delta f(s, \Pi_s^1)) ds \\ \quad + \int_0^t (\sigma_x^0(s, X_{s-}^\varepsilon, Y_{s-}^\varepsilon) \nabla X_s^\varepsilon + \sigma_y^0(s, X_{s-}^\varepsilon, Y_{s-}^\varepsilon) \nabla Y_s^\varepsilon + \Delta \sigma(s, \Pi_s^1)) dH_s; \\ \nabla Y_t^\varepsilon = \varphi_x^0(X_T^\varepsilon) + \Delta \varphi(X_T^1) + \int_t^T (g_x^0(s, \Pi_s^\varepsilon) \nabla X_s^\varepsilon + g_y^0(s, \Pi_s^\varepsilon) \nabla Y_s^\varepsilon + g_z^0(s, \Pi_s^\varepsilon) \nabla Z_s^\varepsilon + \Delta g(s, \Pi_s^1)) ds \\ \quad - \int_t^T \nabla Z_s^\varepsilon dH_s; \end{cases}$$

Then by Theorem 3.1, the above FBSDEs has a unique solution. Moreover, a simple calculation shows that

$$\Delta \Pi_t = \int_0^1 \frac{d}{d\varepsilon} \Pi_t^\varepsilon d\varepsilon = \int_0^1 \nabla \Pi_t^\varepsilon d\varepsilon.$$

since (f^0, σ^0, g^0) satisfies (4.12), by Lemma 4.2, we obtain

$$\|\Delta \Pi^\varepsilon\|^2 \leq CE \left\{ |\Delta X_0|^2 + |\Delta \varphi(X_T^1)|^2 + \int_0^T [|\Delta f|^2 + |\Delta \sigma|^2 + |\Delta g|^2](t, \Pi_t^1) dt \right\},$$

which implies the desired result. \square

Proof of Corollary 3.1 Using Theorem 3.3 we have

$$\begin{aligned} \|\Pi^n - \Pi^n\|^2 &\leq CE \left\{ |X_0^n - X_0^0|^2 + |\varphi^n - \varphi^0|^2(X_T^0) \right. \\ &\quad \left. + \int_0^T [|f^n - f^0|^2 + |\sigma^n - \sigma^0|^2 + |g^n - g^0|^2](t, \Pi_t^0) dt \right\}. \end{aligned}$$

Thus, the desired result follows immediately, by letting n tend to 0, and using the dominated convergence theorem. \square

4.4 Proof of comparison theorem

4.4.1 Some auxiliary results

In order to prove Proposition 3.2, we need the following two Lemmas. Let us introduce the following linear FBSDE

$$\begin{cases} X_t = \int_0^t (\bar{a}_s^1 X_s + \bar{b}_s^1 \bar{Y}_s + \bar{c}_s^1 \bar{Z}_s) ds + \int_0^t (\bar{a}_s^2 X_s + \bar{b}_s^2 \bar{Y}_s) dH_s, \\ \bar{Y}_t = \int_t^T (\bar{a}_s^3 X_s + \bar{b}_s^3 \bar{Y}_s + \bar{c}_s^3 \bar{Z}_s) ds - \int_t^T \bar{Z}_s dH_s. \end{cases} \quad (4.20)$$

Here, $\bar{Y}_t \triangleq Y_t - P_t X_t$, $\bar{Z}_t \triangleq Z_t - P_t (a_t^2 X_t + b_t^2 Y_t) - g_t X_t$, where $P = E(P) + \int_0^T p_t dH_t$, $P_t \triangleq E(P) + \int_0^t p_t dH_t$; and

$$\begin{cases} \bar{a}_t^1 \triangleq a_t^1 + P_t b_t^1 + P_t a_t^2 c_t^1 + |P_t|^2 b_t^2 c_t^1 + p_t c_t^1; \\ \bar{b}_t^1 \triangleq b_t^1 + P_t b_t^2 c_t^1 = b_t^1; \\ \bar{c}_t^1 \triangleq c_t^1; \\ \bar{a}_t^2 \triangleq a_t^2 + P_t b_t^2; \\ \bar{b}_t^2 \triangleq b_t^2; \\ \bar{a}_t^3 \triangleq a_t^3 + p_t a_t^2 + P_t a_t^1 + (b_t^3 + p_t b_t^2 + P_t b_t^1) P_t \\ \quad + (c_t^3 + P_t c_t^1) (p_t + P_t a_t^2 + |P_t|^2 b_t^2); \\ \bar{b}_t^3 \triangleq b_t^3 + p_t b_t^2 + P_t b_t^1 + P_t b_t^2 c_t^3 + |P_t|^2 b_t^2 c_t^1; \\ \bar{c}_t^3 \triangleq c_t^3 + P_t c_t^1. \end{cases}$$

Lemma 4.3. *Let (X, Y, Z) be the solution of LFBSDE (3.1), assume $\beta = 0$ and $p \leq C$. Then $(X, \tilde{Y}, \tilde{Z})$ is the solution of the linear FBSDE (4.20).*

Proof of Lemma 4.3. By the definition of P_t , \bar{Y}_t and \bar{Z}_t , we get

$$\begin{aligned} dX_t &= (a_t^1 X_t + b_t^1 (\bar{Y}_t + P_t X_t) + c_t^1 (\bar{Z}_t + P_t a_t^2 X_t + P_t b_t^2 (\bar{Y}_t + P_t X_t) + p_t X_t)) dt \\ &\quad + (a_t^2 X_t + b_t^2 (\bar{Y}_t + P_t X_t)) dH_t \\ &= (\bar{a}_t^1 X_t + \bar{b}_t^1 \bar{Y}_t + \bar{c}_t^1 \bar{Z}_t) dt + (\bar{a}_t^2 X_t + \bar{b}_t^2 \bar{Y}_t) dH_t, \end{aligned}$$

and

$$\begin{aligned} d\bar{Y}_t &= -(a_t^3 X_t + b_t^3 Y_t + c_t^3 Z_t) dt + Z_t dH_t - p_t (a_t^2 X_t + b_t^2 Y_t) dt \\ &\quad - P_t (a_t^1 X_t + b_t^1 Y_t + c_t^1 Z_t) dt - P_t (a_t^2 X_t + b_t^2 Y_t) dH_t - p_t X_t dH_t \\ &= \bar{Z}_t dH_t - [(a_t^3 + p_t a_t^2 + P_t a_t^1) X_t + (b_t^3 + p_t b_t^2 + G_t b_t^1) (\bar{Y}_t + p_t X_t) \\ &\quad + (c_t^3 + P_t c_t^1) (\bar{Z}_t + (p_t + P_t a_t^2) X_t + P_t b_t^2 (\bar{Y}_t + P_t X_t))] dt \\ &= -(\bar{a}_t^3 X_t + \bar{b}_t^3 \bar{Y}_t + \bar{c}_t^3 \bar{Z}_t) dt + \bar{Z}_t dH_t, \end{aligned}$$

Is easy to prove that $\bar{a}_t^i, \bar{b}_t^i, \bar{c}_t^i$ are bounded and still satisfy the assumptions (4.12). Then this gives the desired result.

Lemma 4.4. *Assume $\alpha = 0$, $c_t^3 = 0$, for some integer m , we assume $\frac{1}{m} \leq \kappa_2 \leq m$. Then there exist small constants δ and C depending on λ and λ_0 , such that $T \leq \delta$, and that for some $\varepsilon > 0$,*

$$\left| E \left(P X_t + \int_0^T (a_t^3 X_t + b_t^3 Y_t) dt \right) \right| \leq C m \sqrt{\varepsilon} T.$$

Proof of Lemma 4.4. By standard arguments and using Young's inequality, for every $\varepsilon > 0$, there exist constant C depending only on λ, λ_0 , that

$$\begin{aligned} \sup_{0 \leq t \leq T} E (|X_t|^2 + |Y_t|^2) + E \left(\int_0^T \|Z_t\|_{l^2(\mathbb{R})}^2 dt \right) &\leq C \varepsilon^{-1} E \left(\int_0^T (|X_t|^2 + |Y_t|^2) dt \right) + \frac{\varepsilon}{2} E \left(\int_0^T |\beta_t|^2 dt \right) \\ &\leq C \varepsilon^{-1} T \sup_{0 \leq t \leq T} E (|X_t|^2 + |Y_t|^2) + \frac{\varepsilon}{2} m^2 T. \end{aligned}$$

If we choose the constant $\delta = \frac{\varepsilon}{2C}$ and will specify ε later. Then for $T \leq \delta$, we get

$$\sup_{0 \leq t \leq T} E (|X_t|^2 + |Y_t|^2) + E \left(\int_0^T \|Z_t\|_{l^2(\mathbb{R})}^2 dt \right) \leq m^2 \varepsilon T.$$

And

$$\begin{aligned} E (|X_t|^2) &\leq C E \left(\left| \int_0^T (a_t^1 X_t + b_t^1 Y_t + c_t^1 Z_t) dt \right|^2 + \left| \int_0^T (a_t^2 X_t + b_t^2 Y_t) dH_t \right|^2 \right) \\ &\leq C E \left(T \int_0^T (|X_t|^2 + |Y_t|^2 + \|Z_t\|_{l^2(\mathbb{R})}^2) dt + \int_0^T (|X_t|^2 + |Y_t|^2) dt \right) \\ &\leq C m^2 \varepsilon T^2. \end{aligned}$$

Thus

$$\begin{aligned} &\left| E (P X_t) + \int_0^T (a_t^3 X_t + b_t^3 Y_t) dt \right| \\ &\leq C E^{\frac{1}{2}} (|X_T|^2) + C T \sup_{0 \leq t \leq T} E^{\frac{1}{2}} (|X_t|^2 + |Y_t|^2) \leq C m \sqrt{\varepsilon} T. \end{aligned}$$

This ends the proof. \square

4.4.2 Proof of Proposition.3.2.

The proof of the proposition 3.2 will be splitted into several steps.

Step 1. Assume that $P = 0$ and $\beta = 0$. If $Y_0 < 0$, let us define the following stopping time

$$\tau \triangleq \inf \{t : Y_t = 0\} \wedge T.$$

Since $Y_T = \alpha \geq 0$, we get $Y_\tau = 0$. Define

$$\begin{aligned} \hat{a}_t^i &\triangleq a_t^i 1_{\{\tau > t\}}; \hat{b}_t^i \triangleq b_t^i 1_{\{\tau > t\}}; \hat{c}_t^i \triangleq c_t^i 1_{\{\tau > t\}} \\ \hat{X}_t &\triangleq X_{\tau \wedge t}; \quad \hat{Y}_t \triangleq Y_{\tau \wedge t}; \quad \hat{Z}_t \triangleq Z_{\tau \wedge t} \end{aligned}$$

In view of Lemma 4.2, the following LFBSDE:

$$\begin{cases} \hat{X}_t = \int_0^t \left(\hat{a}_s^1 \hat{X}_s + \hat{b}_s^1 \hat{Y}_s + \hat{c}_s^1 \hat{Z}_s \right) ds + \int_0^t \left(\hat{a}_s^2 \hat{X}_s + \hat{b}_s^2 \hat{Y}_s \right) dH_s, \\ \hat{Y}_t = \int_t^T \left(\hat{a}_s^3 \hat{X}_s + \hat{b}_s^3 \hat{Y}_s + \hat{c}_s^3 \hat{Z}_s \right) ds - \int_t^T \hat{Z}_s dH_s, \end{cases}$$

has a unique solution, with $\hat{Y}_T = 0$. That is to say $Y_0 = \hat{Y}_0 = 0$, obviously this leads to a contradiction. In other words, we have proved that $Y_0 \geq 0$.

Step 2. Assume that all the conditions in Lemma 4.3 are fulfilled, then $\bar{Y}_T = \alpha \geq 0$. Applying *Step 1* we get $Y_0 = \hat{Y}_0 \geq 0$.

Step 3. Assume $\beta = 0$. One can find P_n satisfying the condition in Lemma 4.3 such that $P_n \rightarrow P$ a.s. and $|P_n| \leq \lambda$. Let (X^n, Y^n, Z^n) denotes the solution corresponding to G_n . Apply the result of *Step 2* to conclude that $Y_0^n \geq 0$. Then from Corollary 3.1, we get $Y_0 = \lim_{n \rightarrow \infty} Y_0^n \geq 0$.

Step 4. Assume all the conditions in Lemma 4.4 are in force. Then

$$\begin{aligned} Y_0 &= E \left(PX_T + \int_0^T (a_t^3 X_t + b_t^3 Y_t + \beta_t) dt \right) \\ &\geq m^{-1}T - \left| E \left(PX_T + \int_0^T (a_t^3 X_t + b_t^3 Y_t) dt \right) \right| \\ &\geq m^{-1}T - Cm\sqrt{\varepsilon}T. \end{aligned}$$

Now choose $\varepsilon = C^{-2}m^{-4}$, we get $Y_0 \geq 0$.

Step 5. Assume $\frac{1}{m} \leq \beta \leq m$ and $T \leq \delta$, where δ is the same as in Lemma 4.4. Denote

$$\begin{cases} X'_t = \int_0^t (a_s^1 X'_s + b_s^1 Y'_s + c_s^1 Z'_s) ds + \int_0^t (a_s^2 X'_s + b_s^2 Y'_s) dH_s, \\ Y'_t = PX'_T + \alpha + \int_t^T (a_s^3 X'_s + b_s^3 Y'_s + c_s^3 Z'_s) ds - \int_t^T Z'_s dH_s, \end{cases}$$

and

$$\begin{cases} X''_t = \int_0^t (a_s^1 X''_s + b_s^1 Y''_s + c_s^1 Z''_s) ds + \int_0^t (a_s^2 X''_s + b_s^2 Y''_s) dH_s, \\ Y''_t = LX''_T + \int_t^T (a_s^3 X''_s + b_s^3 Y''_s + c_s^3 Z''_s + \beta_s) ds - \int_t^T Z''_s dH_s, \end{cases}$$

By Step 3, $Y'_0 \geq 0$, and by Step 4, $Y''_0 \geq 0$. Then, $Y_0 = Y'_0 + Y''_0 \geq 0$.

Step 6. Assume $\frac{1}{m} \leq \beta \leq m$. Let δ be as in Lemma 4.4 but corresponding to $(\lambda, \bar{\lambda}_0, m)$ instead of (λ, λ_0, m) , and assume $(n-1)\delta < T < n\delta$. Denote $T_i \triangleq \frac{iT}{n}$, $L_n \triangleq L$ and $\alpha_n \triangleq \alpha$. For $t \in [T_{n-1}, T_n]$, let

$$\begin{cases} X_t^{n,1} = 1 + \int_{T_{n-1}}^t (a_s^1 X_s^{n,1} + b_s^1 Y_s^{n,1} + c_s^1 Z_s^{n,1}) ds + \int_{T_{n-1}}^t (a_s^2 X_s^{n,1} + b_s^2 Y_s^{n,1}) dH_s, \\ Y_t^{n,1} = P_n X_T^{n,1} + \int_t^{T_n} (a_s^3 X_s^{n,1} + b_s^3 Y_s^{n,1} + c_s^3 Z_s^{n,1}) ds - \int_t^{T_n} Z_s^{n,1} dH_s, \end{cases}$$

and

$$\begin{cases} X_t^{n,2} = 1 + \int_{T_{n-1}}^t (a_s^1 X_s^{n,2} + b_s^1 Y_s^{n,2} + c_s^1 Z_s^{n,2}) ds + \int_{T_{n-1}}^t (a_s^2 X_s^{n,2} + b_s^2 Y_s^{n,2}) dH_s, \\ Y_t^{n,2} = P_n X_T^{n,2} + \alpha_n + \int_t^{T_n} (a_s^3 X_s^{n,2} + b_s^3 Y_s^{n,2} + c_s^3 Z_s^{n,2} + \beta_s) ds - \int_t^{T_n} Z_s^{n,2} dH_s, \end{cases}$$

Denote

$$P_{n-1} \triangleq Y_{T_{n-1}}^{n,1}, \alpha_{n-1} \triangleq Y_{T_{n-1}}^{n,2}.$$

By the proof of Theorem 3.2, we know that $|P_{n-1}| \leq \lambda_1 \leq \bar{\lambda}_0$. Apply the result of *Step 5*, we get $\alpha_{n-1} \geq 0$. We note that, for $t \in [0, T_{n-1}]$, (X, Y, Z) satisfies

$$\begin{cases} X_t = \int_0^t (a_s^1 X_s + b_s^1 Y_s + c_s^1 Z_s) ds + \int_0^t (a_s^2 X_s + b_s^2 Y_s) dH_s, \\ Y_t = P_{n-1} X_{T_{n-1}} + \alpha_{n-1} + \int_t^{T_{n-1}} (a_s^3 X_s + b_s^3 Y_s + c_s^3 Z_s + \beta_s) ds - \int_t^{T_{n-1}} Z_s dH_s. \end{cases}$$

Repeating the same arguments, we may define L_1 and $\alpha_1 \geq 0$, and it holds that

$$\begin{cases} X_t = \int_0^t (a_s^1 X_s + b_s^1 Y_s + c_s^1 Z_s) ds + \int_0^t (a_s^2 X_s + b_s^2 Y_s) dH_s, \\ Y_t = P_1 X_{T_1} + \alpha_1 + \int_t^{T_1} (a_s^3 X_s + b_s^3 Y_s + c_s^3 Z_s + \beta_s) ds - \int_t^{T_1} Z_s dH_s. \end{cases}$$

By step 5, we have $Y_0 \geq 0$.

Step 7. In the general case, we put $\beta^m \triangleq (\beta \wedge m) \vee \frac{1}{m}$ and let (X^m, Y^m, Z^m) denote the solution corresponding to β^m . We know by Step 6, that $Y_0^m \geq 0$. Then by Corollary 3.1, $Y_0 = \lim_{m \rightarrow \infty} Y_0^m \geq 0$. This gives the result. \square

We are now in position to give the proof of comparison theorem.

4.4.3 Proof of Theorem 3.4.

For $0 \leq \varepsilon \leq 1$, let Π^ε and $\nabla \Pi^\varepsilon$ be as in the prove of Theorem 3.3. Then, we get $\Delta X_0 = 0, \Delta f = 0, \Delta \sigma = 0, \Delta g \geq 0, \Delta \varphi \geq 0$. From Proposition 3.2, we have $\nabla Y_0^\varepsilon \geq 0$. This proves the theorem. \square

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